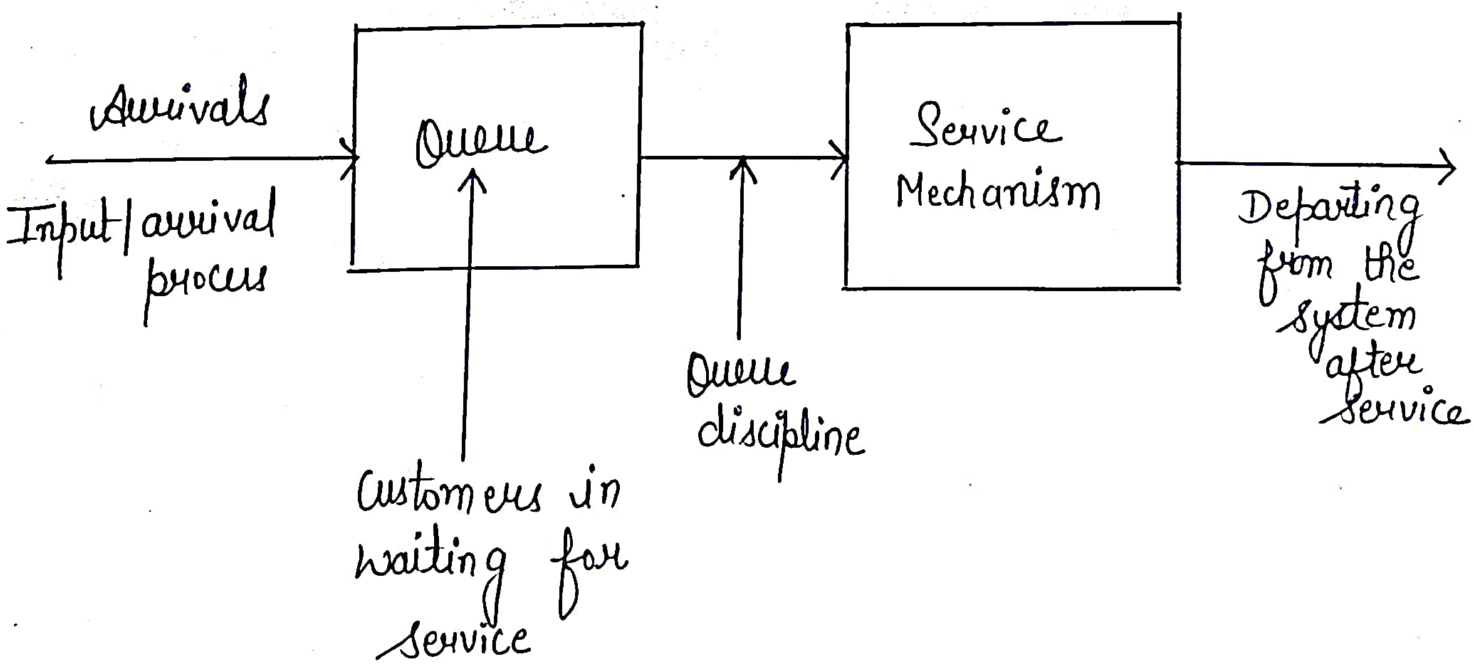
Class - Msc. (Mathematics)
Subject - Operation Research II

Topic :- Queueing Theory

Queue :- A group of items waiting to receive service including those receiving the service is known as a Queue or a waiting line.

Queuing System | Process :-



Queue Discipline

- FIFO (First in first out)
- LIFO (Last in first out)
- SIRO (Service in random order)

Other Terminologies :-

Queue length, Idle time, Busy time, Arrival rate (λ), Service rate (μ)

States of the system :-

- **Transient state :-** A system is said to be in transient state if the operating characteristics are dependent upon time.
- **Steady state :-** A system is said to be in steady state when the operating characteristics are Independent of time.

Let $P_n(t)$ be the probability of n customers in the system at any time t, then the system acquires a steady state as $t \to \infty$ if

$$\lim_{t \to \infty} P_n(t) = P_n \quad \text{(i.e independent of } t\text{)}.$$

- **Explosive state :-** If the arrival rate of the system is more than the servicing rate then, the length of the Queue goes on increasing and increasing with time and will tend to infinity as $t \to \infty$. This state of the system is called Explosive state.

Poisson Process :-

In Poisson process the prob. of n arrivals during time interval of length t is given by

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{\lfloor n} \quad \text{—— (1)}$$

where λt is the parameter.

Case I :- When $n=0$

{ There is no customer in the system in small interval Δt }

$$P_0(\Delta t) = \frac{(\lambda(\Delta t))^0 e^{-\lambda \Delta t}}{\lfloor 0}$$

$$= \frac{1 \cdot e^{-\lambda \Delta t}}{1}$$

$$= e^{-\lambda \Delta t}$$

$$= 1 - \lambda \Delta t + \frac{\lambda^2 (\Delta t)^2}{2!} - \frac{\lambda^3 (\Delta t)^3}{3!} + \cdots$$

{ By using Expansion of $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$ }

$$= 1 - \lambda \Delta t + \frac{\lambda^2 (\Delta t)^2}{2!} - \frac{\lambda^3 (\Delta t)^3}{3!} + \cdots$$

$$= 1 - \lambda \Delta t + o(\Delta t)$$

where $o(\Delta t)$ is a quantity of small order and if Δt is very small then $o(\Delta t)=0$

In this case, $\boxed{P_0(\Delta t) = 1-\lambda\Delta t}$

Case-II when $n=1$

{1 arrival occur in the system in time Δt}

$$P_1(\Delta t) = \frac{(\lambda(\Delta t))^1 e^{-\lambda\Delta t}}{\lfloor 1}$$

$$= \frac{\lambda(\Delta t)\, e^{-\lambda\Delta t}}{1}$$

$$= \lambda(\Delta t)\, e^{-\lambda\Delta t}$$

$$= \lambda\Delta t \left[1 - \frac{\lambda\Delta t}{\lfloor 1} + \frac{(\lambda\Delta t)^2}{\lfloor 2} - \cdots\right]$$

$$= \lambda\Delta t - (\lambda\Delta t)^2 + (\text{order of } \Delta t)$$

$$= \lambda\Delta t + o(\Delta t)$$

$$\boxed{P_1(\Delta t) = \lambda\Delta t}$$

Case-III when $n=m$ but greater than 1

i.e when $n = m > 1$

Putting $n=m$ in eqn (1), we get

$$P_m(\Delta t) = \frac{(\lambda\Delta t)^m e^{-\lambda\Delta t}}{\lfloor m}$$

$$= \frac{(\lambda \Delta t)^m \left[1 - \frac{\lambda \Delta t}{1!} + \frac{(\lambda \Delta t)^2}{2!} - \dots\right]}{\lfloor m}$$

$$= \frac{\lambda^m (\Delta t)^m}{m!} \left[1 - \frac{\lambda \Delta t}{1!} + \frac{(\lambda \Delta t)^2}{2!} - \dots\right]$$

$$= \frac{\lambda^m}{m!} \left[(\Delta t)^m - \frac{\lambda (\Delta t)^m (\Delta t)}{1!} + \frac{\lambda^2 (\Delta t)^m (\Delta t)^2}{2!} - \dots\right]$$

$$= \frac{\lambda^m}{m!} \left[(\Delta t)^m - \lambda \frac{(\Delta t)^{m+1}}{1!} + \lambda^2 \frac{(\Delta t)^{m+2}}{2!} - \dots\right]$$

$$= \frac{\lambda^m}{m!} \left[(\Delta t)^m - \lambda (\Delta t)^{m+1} + \dots\right]$$

If Δt is very small, then

$$(\Delta t)^m - (\Delta t)^{m+1} + \dots = 0$$

then $\boxed{P_m(\Delta t) = 0}$

There are three axioms which a process should follow as to become a Poisson process and the corresponding queue is known as Poisson queue.